ON THE LOCATION OF THE ROOTS OF THE JACOBIAN OF TWO BINARY FORMS, AND OF THE DERIVATIVE OF A RATIONAL FUNCTION*

BY

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The present paper is an extension and in some respects a simplification of a recent paper published under the same title.† Both papers are based on a theorem (Theorem I, below) due to Professor Bôcher.‡ By means of the statical problem of determining the positions of equilibrium in a certain field of force, there are obtained some new results concerning the location of the roots of the jacobian of two binary forms relative to the location of the roots of the ground forms. Application is made to the roots of the derivative of a polynomial and to the roots of the derivative of a rational function. The present paper gives a proof and an application of a geometrical theorem (Theorem II) which may be not uninteresting.

Bôcher considers a number of fixed particles in a plane or by stereographic projection on the surface of a sphere, and supposes each particle to repel with a force equal to its mass (which may be positive or negative) divided by the distance. If the plane is taken as the Gauss plane, the following result is proved:§

THEOREM I. *The vanishing of the jacobian of two binary forms* f_1 *and* f_2 *of degrees* p_1 *and* p_2 *respectively determines the points of equilibrium in the field of force due to* p_1 *particles of mass* p_2 *situated at the roots of* f_1, *and* p_2 *particles of mass* $-p_1$ *situated at the roots of* f_2.

The jacobian vanishes not only at the points of no force, but also at the multiple roots of either form or a common root of the two forms; such a point is called a position of *pseudo-equilibrium.*

* Presented to the Society, Dec. 31, 1919.

† Walsh, these Transactions, vol. 19 (1918), pp. 291–298. This paper will be referred to as I.

‡ Maxime Bôcher, *A problem in statics and its relation to certain algebraic invariants*, Proceedings of the American Academy of Arts and Sciences, vol. 40 (1904), p. 469.

§ Bôcher's proof (l. c., p. 476) is reproduced in I, p. 291.

It is intuitively obvious that there can be no position of equilibrium very near any of the fixed particles, or very near and outside of a circle containing a number of fixed particles, all attracting or all repelling, if the other particles are sufficiently remote. We consider, then, a number of particles in a circle or more generally in a circular region. First we adjoin to the plane the point at infinity, and use the term *circle* to include point and straight line; then we define a *circular region* to be a closed region of the plane bounded by a circle, namely, the interior of a circle, the exterior of a circle including the point at infinity, a half plane, a point, or the entire plane. There will be no confusion in having the same notation for a circular region as for its boundary.

In the following development we shall use several lemmas.

LEMMA I. *The force at a point P due to k particles each of unit mass situated in a circular region C not containing P is equivalent to the force at P due to k coincident particles each of unit mass also in C.*

Denote by C' the inverse of C in the circle of unit radius and center P and by Q' the inverse of any point Q with regard to that circle. The force at P due to a particle at Q is in direction and magnitude PQ'. We replace k vectors PQ' by k coincident vectors having one terminal at P and the other at the center of gravity of the points Q'; these two sets of vectors have the same resultant. If any point Q is in the region C, its inverse Q' is in C', and the center of gravity of a number of such points Q' is also in C'. The inverse of this center of gravity is then in C.

LEMMA II. *In the field of force due to k positive particles at z_1, l positive particles at z_2, and $k + l$ negative particles at z_3, the only position of equilibrium is z_4 as determined by the cross-ratio*

$$\frac{(z_1 - z_2)(z_3 - z_4)}{(z_2 - z_3)(z_4 - z_1)} \equiv (z_1, z_2, z_3, z_4) = \frac{k + l}{k}.$$

The lemma is evidently true when one of the points z_1, z_2, z_3 is at infinity. The invariance of the positions of equilibrium under linear transformation follows from Theorem I and hence completes the proof.

We shall next prove a preliminary theorem, the proof of which is given in part by several succeeding lemmas.

THEOREM II. *If the envelopes of points z_1, z_2, z_3 are circular regions C_1, C_2, C_3 respectively, then the envelope of z_4, defined by the real constant cross-ratio*

$$\lambda = (z_1, z_2, z_3, z_4)$$

*is also a circular region.**

* The term *envelope* is used to denote the set of points which is the totality of positions assumed by each of the points z_1, z_2, z_3, z_4; the points z_1, z_2, z_3 are supposed to vary independently.

The proof of Theorem II which is presented in detail has some advantages and some dis-

We denote the envelope of z_4 by C_4, and we must show that C_4 is a region bounded by a single circle. First we consider several special cases of the theorem. If C_1, C_2, and C_3 are distinct points, C_4 is a point. If any of the regions C_1, C_2, C_3 is the entire plane, C_4 is also the entire plane. If $\lambda = 0$ and if C_1 and C_2 have a point in common, C_4 is the entire plane. If $\lambda = 0$ and C_1 and C_2 have no point in common, $z_3 = z_4$ and so C_4 coincides with C_3. If $\lambda = \infty$ and C_2 and C_3 have a common point, C_4 is the entire plane. If $\lambda = \infty$ and C_2 and C_3 have no common point, C_4 and C_1 are identical. If $\lambda = 1$ and C_1 and C_3 have a common point, C_4 is the entire plane. If $\lambda = 1$ yet C_1 and C_3 have no common point, C_4 is identical with C_2. In the sequel, unless it is explicitly stated to the contrary, we suppose λ to have none of the values $0, 1, \infty$. It follows that no two of the points z_1, z_2, z_3, z_4 coincide unless three of them coincide.

Except in the trivial case that C_1, C_2, C_3 are points, C_4 is evidently a two-dimensional continuum and is not necessarily the entire plane. The envelope C_4 is connected, for to join any pair of points z_4', z_4'' in C_4 by a curve in C_4, we need merely to choose any set of points corresponding to each, z_1', z_2', z_3'; z_1'', z_2'', z_3'', in the proper regions. Join z_1' and z_1'' by a continuous curve which lies in C_1, and similarly join z_2' and z_2'', and z_3' and z_3'', by continuous curves in C_2 and C_3 respectively. Allow z_1, z_2, z_3 to move from z_1', z_2', z_3' to z_1'', z_2'', z_3'' along these respective curves. The point z_4 corresponding moves from z_4' to z_4'' in C_4 and along a curve which is continuous because z_4 is a linear function of z_1, z_2, z_3.

Our next remark is stated explicitly as a lemma. It is readily stated and established for regions whose boundaries are curves much more general than circles, but we consider here merely the form under the hypothesis of Theorem II and for application to the proof of that theorem.

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advantages over the following suggested method of proof. The theorem is evidently true when C_1, C_2, and C_3 are points. The theorem is easily proved when C_1 and C_2 are points but C_3 is not a point. By taking the envelope of the circular region C_4 in the preceding degenerate case, the theorem can be proved when C_1 is a point but neither C_2 nor C_3 is a point. The envelope of the region C_4 in this last degenerate case, as z_1 is allowed to vary over a region C_1 not a point, gives the envelope of z_4 for the theorem in its generality. I have not been able to carry through the actual analytic determination of the envelope by this method because the algebraic work is too laborious.

This suggested method of proof, however, shows at once that the boundary of the region C_4 in the general case is an algebraic curve or at least part of an algebraic curve.

It seems to me likely that Theorem II is true also when λ is imaginary, but I have not carried through the proof in detail.

In general the relation of the regions C_1, C_2, C_3, C_4 is not reciprocal. For example if C_1 is a point but neither C_2 nor C_3 is a point and if these regions lead to the fourth region C_4, then if we choose the circular regions C_2, C_3, C_4 as the original circular regions of the lemma, we cannot for any choice of λ be led to the region C_1. This lack of reciprocality does not depend on the degeneracy of one of the regions C_1, C_2, C_3, C_4.

LEMMA III. *If the point z_4 is on but not at a vertex of the boundary of C_4,* then any set of points z_1, z_2, z_3 corresponding lie on the boundaries of the respective regions C_1, C_2, C_3; the circle C through the points z_1, z_2, z_3, z_4 cuts the circles C_1, C_2, C_3 all at angles of the same magnitude; and if C is transformed into a straight line, the lines tangent to the circles C_1, C_2, and C_3 at the points z_1, z_2, z_3 respectively are parallel.*

The following proof is formulated only for the general case that none of the circles C_1, C_2, C_3 is a null circle, but no essential modification is necessary to include the degenerate cases.

When z_2 and z_3, and also the circle C are kept fixed, a continuous motion of z_1 along C also causes z_4 to move continuously along C. If the direction of motion of z_1 is reversed, the direction of motion of z_4 is also reversed. Hence z_4 is not on the boundary of C_4 unless z_1 is on the boundary of C_1, and as can be shown in an analogous manner, not unless z_2 and z_3 are on the boundaries of C_2 and C_3 respectively. The region C_4 is closed since the regions C_1, C_2, and C_3 are closed.

Let P be any point of the boundary of C_4. Transform P to infinity, so that the corresponding points z_1, z_2, z_3 lie on the same line L. We assume at first that L is not tangent to any of the circles C_1, C_2, C_3 nor to the boundary of C_4. The relative positions of the points z_1, z_2, z_3 on L together with the sense along L in which the region C_1 extends from z_1 determine uniquely the sense along L in which the regions C_2, C_3, C_4 must extend from z_2, z_3, P respectively. There is evidently a segment of L terminated by P composed entirely of points in C_4. If the lines tangent to the circles C_2 and C_3 at the points z_2 and z_3 are not parallel, it is possible slightly to rotate L about z_1 in one direction or the other into a new position L' and to determine a point z_2'' on L' and on the circle C_2 and a point z_3'' on L' and *interior to the region* C_3 such that the triangles $z_1 z_2 z_2''$ and $z_1 z_3 z_3''$ are similar and hence we have the relation

$$(z_1, z_2'', z_3'', P) = \lambda.$$

Then z_3'' can be moved in either sense along the line L' and still remain in its proper envelope, so there are corresponding points z_4'' on L' in either sense from P. Moreover, this is true for every position of L' if the angle from L to L' is in the proper sense and is sufficiently small, so if we transform P to the finite part of the plane and z_1 to infinity and notice that the lines L' are lines through the point P, it becomes evident that there are points z_4 in the neighborhood of P on any line L' through P which lies within a certain sector whose vertex is P, and there are points z_4 on L' in both directions

* It is of course true that the boundary of C_4 has no vertices, but that fact has not yet been proved.

from P. Hence if P is actually on the boundary of C_4, it must lie at a vertex of that boundary.*

The proof thus far has been formulated to prove that when P is at infinity the lines tangent to the circles C_2 and C_3 at z_2 and z_3 are parallel. The notation of the proof can easily be modified to show that the lines tangent to the circles C_1 and C_2 at z_1 and z_2 are parallel, and hence the lines tangent to C_1, C_2, C_3 at z_1, z_2, z_3 are parallel.

This same method of reasoning is readily used to prove that if the circle C of the lemma is tangent to one or two of the circles C_1, C_2, C_3 at the respective points z_1, z_2, z_3 but is not tangent to all these circles, the boundary of C_4 has a vertex at z_4. The circle C is not tangent to the boundary of C_4 unless C is tangent to C_1, C_2, and C_3. This consideration completes the proof of Lemma III.

It is desirable to make a revision in our use of the term *angle between two circles*. With Coolidge,† we consider circles to be described by a point moving in a counter-clockwise sense, and define the angle between two circles to be the angle between the half-tangents drawn at the intersection in the sense of description of the circles. When we are concerned with a single straight line, either sense may be given to it. We shall use this convention in proving the following lemma, which is a result purely of circle geometry which has not necessarily any connection with Theorem II. As stated and proved, it is slightly more general than is necessary for its application in the proof of that theorem.

LEMMA IV. *Suppose a variable circle C either to cut three distinct fixed non-coaxial circles C_1, C_2, C_3 all at the same angle or to cut a definite one of those circles at an angle supplementary to the angle cut on the other two. If the points z_1, z_2, z_3 are chosen as intersections of C with C_1, C_2, C_3 respectively such that when C is transformed into a straight line the lines tangent to C_1, C_2, C_3 at z_1, z_2, z_3 are all parallel, then the locus of the point z_4 defined by the real constant cross-ratio*

$$\lambda = (z_1, z_2, z_3, z_4)$$

is a circle C_4 which is also cut by C at an angle equal or supplementary to the angles cut on C_1, C_2, C_3.‡

This lemma is not true if the circles C_1, C_2, C_3 are coaxial circles having no point in common. For transform these circles into concentric circles. Then

* The method of proof used in this paragraph was suggested to me by Professor Birkhoff.

† *A treatise on the circle and the sphere*, p. 108.

‡ We remark that the circle C_4 can be constructed by ruler and compass whenever λ is rational or in fact whenever λ is given geometrically. For the circle C can be constructed by ruler and compass in any position; cf. Coolidge, l. c., p. 173. Hence we can determine any number of sets of points z_1, z_2, z_3 and therefore construct any number of points z_4, which enables us to construct C_4.

the circle C is a straight line orthogonal to these circles, C has two intersections with each, and on any particular circle C the points z_1, z_2, z_3 may be chosen on their proper circles so as to lead to four circles of type C_4, in general distinct, and concentric with C_1, C_2, C_3. All these four circles of type C_4 form the locus of points z_4. The situation is essentially the same if C_1, C_2, C_3 are coaxial circles having two common points; we are led to four circles C_4 which are in general distinct. But if we suppose C to vary continuously and also the points z_1, z_2, z_3, z_4 each to vary in one sense continuously, although of course we allow these points to go to infinity but not to occupy any position more than once, the lemma is true even for coaxial circles having no point or two points in common. These situations are included in the detailed treatments given under Cases I and II below.

This lemma breaks down also if the circles C_1, C_2, C_3 are coaxial circles all tangent at a single point, for we can consider the three points z_1, z_2, z_3 to coincide at that point; any circle C through that point satisfies the conditions of the lemma, any point of C can be chosen as z_4, whence it appears that the locus of z_4 is then the entire plane. But if we make not only our previous convention but in addition the convention that not all of the points z_1, z_2, z_3 shall lie at a point common to the three circles unless the fourth point coincides with them, then the lemma remains true. This situation is treated in detail under Case IV below.

The lemma is true but trivial in the degenerate cases $\lambda = 0, 1$, or ∞, for in these cases z_4 coincides with z_3, z_2, or z_1 respectively. The case that C_1, C_2, and C_3 are all null circles is likewise trivial. In the consideration of other cases we shall use the following theorem:

THEOREM. *If three circles be given not all tangent at one point, the circles cutting them at equal angles form a coaxial system, as do those cutting one at angles supplementary to the angles cut on the other two.**

Then as the circle C of Lemma IV varies, it always belongs to a definite coaxial system, unless C_1, C_2, C_3 are all tangent at a single point. This system may consist of (Case I) circles through two points, (Case II) non-intersecting circles, or (Case III) circles tangent to a line at a single point. Under Case IV will be treated the situation when C_1, C_2, C_3 are all tangent at a point. We consider these cases in order.

In Case I, transform to infinity one of the two points through which the coaxial family C passes, so that this family becomes the straight lines through a finite point q of the plane. In general q will be a center of similitude for each pair of the circles C_1, C_2, and C_3. These circles may or may not surround q.

* This statement differs from that of Coolidge, l. c., p. 111, Theorem 219, for we have adjoined to the plane the point at infinity. Theorem 220 seems to be erroneous; compare the four circles C_1, C_2, C_3, C_4 of Lemma IV.

Let z_4 be any point corresponding to the points z_1, z_2, z_3 on C_1, C_2, C_3 respectively. These four points lie on the line qz_4, and we have supposed that the lines tangent to C_1, C_2, C_3 at the points z_1, z_2, z_3 are parallel. Then when the line qz_4 (that is, the circle C) rotates about q, it will be seen that the point z_4 as determined by its constant cross-ratio with z_1, z_2, z_3 will trace a circle C_4 such that q is a center of similitude for any of the pairs of circles C_1, C_2, C_3, C_4. If these circles do not surround q, they have two common tangents belonging to the family C, and the properly chosen cross-ratio of the points of tangency is λ. If C_1, C_2, and C_3 are coaxial, C_4 is coaxial with them. Perhaps it is worth noticing that any circle C_4 such that q is a center of similitude for any pair of the circles C_1, C_2, C_3, C_4 is the circle C_4 of the lemma for a proper choice of λ; in particular C_4 may be the point q or the point at infinity.

Under Case I there are some special situations to be included. If one or more of the circles C_1, C_2, C_3 passes through q, then each of the other circles if not a null circle either is tangent to that circle at q or is a line parallel to the line tangent to that circle at q. If two of the original circles, for definiteness C_1 and C_2, are tangent at q and the other circle C_3 is a line parallel to their common tangent at q, then either z_4 coincides with z_1 and z_2 at q, or z_3 remains at infinity during the motion of C while z_4 traces a circle coaxial with C_1 and C_2; in particular this circle C_4 may be the null circle q. The four circles C_1, C_2, C_3, C_4 have a common tangent circle, namely the line tangent to C_1, C_2, C_4 at q. In the case just considered, one of the circles which passes through q, for definiteness C_1, may be tangent at q to the second circle C_2 which is a straight line. The circle C_3 is a line parallel to C_2. When the circle C varies, z_4 coincides with z_1 and z_2 at q, z_4 coincides with z_2 and z_3 at infinity, or the circle C coincides with C_2, z_1 with q, and z_3 with the point at infinity, while z_2 traces the line C_2 and hence z_4 also traces C_2. The circles C_1, C_2, C_3, C_4 have a common tangent circle C_2. If one of the original circles, for definiteness C_1, passes through q and the circles C_2 and C_3 are lines parallel to the tangent to C_1 and q, then the circle C_4 is a circle coaxial with C_2 and C_3 which may be the point at infinity. The four circles C_1, C_2, C_3, C_4 have as common tangent circle the line tangent to C_1 at q.

The general situation of Case I is not essentially changed and requires no further discussion if one of the circles C_1, C_2, C_3 is a point (q or the point at infinity) or if two of them are points (q and the point at infinity), except when at least one of the null circles lies on one of the non-null circles. In particular, if two circles, for example C_1 and C_2, are null circles and one of them (say C_2) lies on the non-null circle C_3, the locus of z_4 is a circle C_4 tangent to the circle C_3 at the point C_2. If the two null circles C_1 and C_2 both lie on the non-null circle C_3, the circle C is effectually the circle C_3, and C_4 coincides with C_3.

The special situations which we have considered under Case I may similarly degenerate by having one of the original circles a null circle. We shall discuss merely some typical examples. If C_1 and C_2 are tangent at q and C_3 is a null circle at infinity, C_4 is a circle tangent to C_1 and C_2 at q and may be the point q itself. If C_1 is a null circle at q, if C_2 is a circle passing through q, and if C_3 is a line parallel to the tangent to C_2 at q, C_4 is a circle tangent to C_2 at q. If C_1 is a null circle at q, if C_2 is a line passing through q, and C_3 is a line parallel to C_2, then C is essentially the single circle C_2, and C_4 coincides with C_2.

In Case II, the coaxial family C is composed of circles having no point in common, and hence there are two null circles of the family. Transform one of these null circles to infinity, so that the family C becomes a family of circles with a common center p. In the general case, the circles C_1, C_2, and C_3 are all of equal radii and any of them can be brought into coincidence with any other of them by a rotation about p. The point p is outside, on, or within all three circles according as it is outside, on, or within any one of them. Choose any point z_4 of the lemma; then z_1, z_2, z_3, z_4 lie on the circle C whose center is p. As C varies, its radius simply increases or decreases, and z_1, z_2, z_3 rotate about p so that the angles z_2pz_3, z_3pz_1, z_1pz_2 remain constant. Hence z_4 traces a circle C_4 whose radius is equal to the common radius of C_1, C_2, and C_3; moreover any two of the four circles C_1, C_2, C_3, C_4 can be brought into coincidence by a rotation about p. The four circles have two common tangent circles which belong to the family C, one of which may be the point p. The properly chosen cross-ratio of the points of tangency of a tangent circle is λ. Any circle is the circle C_4 of the lemma for a proper choice of λ provided it can be brought into coincidence with any of the circles C_1, C_2, C_3 by a rotation about p.

Another situation that may arise under Case II is that C_1, C_2, and C_3 are straight lines (that is, coaxial circles) through p and the point at infinity; then the locus of z_4 is a circle C_4 coaxial with them. There remains also the possibility that C_1, C_2, C_3 are straight lines all at the same distance from p. Then the circle C_4 is a line also at this same distance from p. There is a circle belonging to the family C which is tangent to C_1, C_2, C_3, C_4, and as before the cross-ratio of the points of contact is λ.

In Case III, the circles C belong to a coaxial family of circles all tangent at a point n, which point we transform to infinity. The circles C become parallel lines and in general C_1, C_2, C_3 become equal circles whose centers are collinear. As C moves parallel to itself, the points z_1, z_2, z_3 remain at equal distances from each other. The locus of z_4 either is a circle C_4 equal to C_1, C_2, and C_3 whose center is collinear with their centers or is the point at infinity. The four circles have two common tangent circles which belong to the family C, and the cross-ratio of the points of tangency of each of these circles is λ.

A degenerate case that should be mentioned is that the point n itself is one of the circles C_1, C_2, C_3. The results are essentially the same as in the general situation. In both the degenerate and the general situations any circle C_4 equal to C_1, C_2, C_3 and whose center is collinear with their centers is the circle C_4 of the lemma if λ is properly chosen.

A special case also occurs if one of the original circles, for definiteness C_1, is a straight line and the other two circles are straight lines parallel to the reflection of C_1 in any of the circles C. When C varies, either z_4 coincides with z_2 and z_3 at infinity, or z_1 is at infinity and z_4 traces a line parallel to C_2 and C_3.

A degenerate case occurs if one of the original circles, say C_3, is the point at infinity, while C_1 and C_2 are the reflections of each other in one of the circles C. Under the conditions of the lemma z_4 must coincide with z_3 at infinity, so C_4 coincides with C_3.

In Case IV, the circles C_1, C_2, C_3 are all tangent at a point m. Transform m to infinity, so that in any non-degenerate case C_1, C_2, C_3 become parallel lines. Under our convention that not all of the points z_1, z_2, z_3 shall lie at m unless z_4 coincides with them, we are led to four circles (in general distinct) according as we allow any one of the points z_1, z_2, z_3 or none of them constantly to lie at infinity. The additional convention already made that z_1, z_2, z_3, z_4 shall vary continuously in one sense and never coincide with any previous position enables us to choose simply one of these circles. The circle C is any straight line, and z_4 is either the intersection of C with a straight line C_4 parallel to C_1, C_2, C_3 or if none of the points z_1, z_2, z_3 is at infinity, z_4 may be constantly the point at infinity. The circles C_1, C_2, C_3, C_4 are all tangent at m.

Under Case IV should be mentioned the degenerate case that one of the circles C_1, C_2, C_3 is a null circle lying at the point of tangency of the other two circles. Our conventions enable us to choose a circle C_4 coaxial with C_1, C_2, C_3.

The proof of Lemma IV is now complete. It will be noticed that except in the special and degenerate cases, the result is entirely symmetric with respect to the four circles C_1, C_2, C_3, C_4. If we commence by choosing any three of those four circles and choose λ properly we shall be led to the other circle. If the last clause in the statement of the lemma is omitted, the lemma is true even if λ is not real.

There is a lemma corresponding to Lemma IV if we suppose two of the original circles, for example C_1 and C_2, to coincide, but suppose C_3 not to coincide with them. If we leave aside the easily treated cases $\lambda = 0, 1$, or ∞, we find either that the points z_1 and z_2 coincide on C_1, in which case z_4 coincides with them and traces the circle C_1, or that if C_1 is a non-null circle z_1 and z_2 do not coincide. In the latter case we are supposing the tangents to C at z_1 and z_2 to be parallel if C is transformed into a straight line and hence C must be orthogonal to C_1 and therefore by the conditions of the lemma also orthogonal to C_3. As before, when the circle C varies it constantly belongs to a definite coaxial system. The reader will easily treat the cases corresponding to Cases I, II, and III above, and also the degenerate case that C_3 is a null circle lying on C_1 and C_2. The results in the general case are quite analogous to the previous results if we notice that C_1, C_2, and C_3 are coaxial. For if C_3 is not a null circle, C cuts C_3 in two distinct points, and by their cross-ratio with z_1 and z_2 these lead to *two distinct circles* C_4 in addition to the circle C_1. Both of these new circles C_4 belong to the coaxial family determined by C_1 and C_3; as C moves it is constantly orthogonal to C_4 as well as to C_1, C_2, C_3. In general, then, the locus of z_4 when C_1 and C_2 coincide is C_1 and two other circles of the coaxial family determined by C_1 and C_3. These two other circles may in a degenerate case coincide, as the reader can easily determine. The convention formerly made, that the points z_1, z_2, z_3, z_4 vary in one sense continuously will of course restrict the locus of z_4 simply to one circle.

When the three circles C_1, C_2, C_3 coincide, we must consider C to coincide with them, or else at least two of the points z_1, z_2, z_3 to coincide and hence z_4 to coincide with them. That is, the circle C_4 corresponding to the circle C_4 of the lemma is the circle C_1.

Lemmas III and IV with the discussion supplementary to the latter do not give us immediately all the material necessary for the proof of Theorem II. For if C_1, C_2, C_3 are coaxial there are four circles, not necessarily all distinct, of the type C_4 of the lemma. If C_1, C_2, C_3 are not coaxial there are also four circles, not necessarily all distinct, of the type C_4 of the lemma, according as C cuts all the circles C_1, C_2, C_3 at equal angles or cuts one at an angle supplementary to the angle cut on the other two. It is conceivable that the boundary of the region C_4 of Theorem II should consist of arcs of more than one distinct circle; we proceed to show that this is in fact never the case.* The following lemma is essential in our proof.

* Whether the boundary of the region C_4 corresponds to motion of C cutting the three original circles at the same angle or a definite one of those circles at an angle supplementary to the angle cut on the other two depends on the relative positions of those circles, on whether the various regions are interior or exterior to their bounding circles, and on the value of λ—in short, on the order of the points z_1, z_2, z_3, z_4 on the circle C. When the regions C_1, C_2, C_3 are mutually external it is easy to prove by reasoning similar to that used in the proof of Lemma III that an arc of only one of the circles of type C_4 can be a part of the boundary of the region C_4. This fact can also be proved in the general case by that same method of reasoning, but the proof given in detail below is perhaps more satisfactory. It is desirable

LEMMA V. *In Theorem II, whenever the envelope of z_4 is not the entire plane, there is a circle S orthogonal to the four circles C_1, C_2, C_3, C_4.*

Whenever the regions C_1, C_2, C_3 have a common point, we may consider z_1, z_2, z_3 to coincide at that point, and consider the cross-ratio of any point z_4 in the plane with those three points to have the value λ, so the envelope of z_4 is the entire plane. In any other case there is a circle S orthogonal to the circles C_1, C_2, C_3. If not every pair of these three original circles intersect, choose two of them which do not intersect, and there will be two points inverse respecting both circles (these points are the null circles of the coaxial family determined by the two circles). Take the inverse of one of those points in the third of the original circles and pass a new circle S through all three points. Then S is orthogonal to the three original circles. If each of the circles C_1, C_2, C_3 has a point in common with the other two, we can transform two of the circles into straight lines (if one of the circles is a null circle the other two circles pass through that null circle and hence the region C_4 is the entire plane). If these two lines are not parallel, the third circle cannot be a straight line nor can it surround the intersection of the other two lines. Hence there is a circle orthogonal to all three circles. If the two lines are parallel the third circle cannot be a straight line. Then there is a circle, in this case a straight line, orthogonal to all three circles. This completes the proof of Lemma V.

Let us transform into a straight line any particular circle S orthogonal to the three original circles and let us suppose not every point of S to be a point of the region C_4; for definiteness assume the point at infinity not to belong to C_4. The positions which each of the three points z_1, z_2, z_3 of Theorem II may occupy fill an entire segment of S, and hence the points z_4 on S which correspond to points z_1, z_2, z_3 on S fill an entire segment of S; we denote this segment by σ. The terminal points of the segment σ are the intersections of S with one of the circles of type C_4 of Lemma IV; we denote that circle by C_4' and the other three circles of that type by C_4'', C_4''', C_4''''. The entire configuration is symmetric with respect to S, so the centers of all the circles C_4', C_4'', C_4''', C_4'''' lie on S. Moreover, S belongs to all four types of circles C of Lemma IV, since it is orthogonal to C_1, C_2, C_3. Hence the intersections of all the circles C_4'', C_4''', C_4'''' are points z_4 which correspond to points z_1, z_2, z_3 lying on S, and hence all those intersections lie on the segment σ. Then of the circles C_4'', C_4''', C_4'''' each is interior to or coincident with C_4'.

Either the entire interior or the entire exterior of each of the circles C_4', C_4'', C_4''', C_4'''' belongs to the region C_4. For the points z_4 which correspond to

that most of the material making up that proof should be given anyway, as a test whether the region C_4 is the entire plane, as giving a ruler-and-compass construction for the circle C_4, and as describing more in detail the entire situation with which we are concerned.

points z_1, z_2, z_3 in the proper regions and on the circle C of Lemma IV fill an entire arc of C, extending from one intersection of C with the circle C_4 to the other intersection. The entire exterior of our circle C_4' does not belong to the region C_4, for the point at infinity does not belong to that region. Hence the entire interior of C_4' does belong to the region C_4. No point external to C_4' can be a point of the boundary of C_4, for none of the circles C_4'', C_4''', C_4'''' has a point exterior to C_4'. Hence the region C_4 is the interior of C_4', under our assumption that not every point of S belongs to the region C_4.

Let us notice that we can allow any or all of the circles C_1, C_2, C_3 to move continuously so as to remain orthogonal to S, so as never to intersect any former position, and so as always to enlarge the regions C_1, C_2, C_3. Then the circle C_4' grows larger and larger, never intersecting its former position, until it becomes the point at infinity, in which case the region C_4 is the entire plane. If the regions C_1, C_2, C_3 are enlarged still further, the region C_4 still remains the entire plane.

Whether or not we assume that not every point of S belongs to the region C_4, we can start with a situation in which not every point of S is a point of C_4 and enlarge the regions C_1, C_2, C_3 in the manner described so as to attain any situation desired in which the region C_4 is not the entire plane. At every stage the region C_4 is a circular region. This completes the proof of Theorem II. We have also obtained a test whether or not the region C_4 is the entire plane. *A necessary and sufficient condition that the region C_4 of Theorem II be the entire plane is that the point z_4 may occupy any position on S and still correspond to points z_1, z_2, z_3 in their proper envelopes and also on S.*

The preceding developments give a comparatively simple ruler-and-compass construction for the circle C_4, whenever λ is rational or is given geometrically. The circle S can be constructed by ruler and compass.* The two points of intersection of S and C_4 can be determined by means of their cross-ratio with properly chosen intersections of S and C_1, C_2, C_3. Since S and C_4 are orthogonal, C_4 can then be constructed.

We shall apply Theorem II in proving our principal theorem.

THEOREM III. *Let f_1 and f_2 be binary forms of degrees p_1 and p_2 respectively, and let the circular regions C_1, C_2, C_3 be the respective envelopes of m roots of f_1, the remaining $p_1 - m$ roots of f_1, and all the roots of f_2. Denote by C_4 the circular region which is the envelope of points z_4 such that*

$$(z_1, z_2, z_3, z_4) = \frac{p_1}{m},$$

when z_1, z_2, z_3 have the respective envelopes C_1, C_2, C_3. Then the envelope of

* Coolidge, l. c., p. 173.

the roots of the jacobian of f_1 and f_2 is the region C_4, together with the regions C_1, C_2, C_3 except that among the latter the corresponding region is to be omitted if any of the numbers $m, p_1 - m, p_2$ is unity. If a region C_i $(i = 1, 2, 3, 4)$ has no point in common with any other of those regions which is a part of the envelope of the roots of the jacobian, it contains of those roots precisely $m - 1$, $p_1 - m - 1, p_2 - 1$, or 1 according as $i = 1, 2, 3$, or 4.

We shall first show by the aid of Lemmas I and II and of Theorems I and II that no point not in C_1, C_2, C_3, or C_4 can be a root of the jacobian. For if a point z_4 is not in C_1, C_2, or C_3 and is a root of the jacobian, it is a position of equilibrium and not of pseudo-equilibrium. The force at z_4 will not be changed if we replace the particles in each of the regions C_1, C_2, C_3 by the same number of coincident particles at points z_1, z_2, z_3 in the respective regions. Then z_4 is a position of equilibrium in the new field of force and hence by Lemma II we have

$$(z_1, z_2, z_3, z_4) = \frac{p_1}{m},$$

and therefore z_4 lies in C_4.

Any point in C_4 can be a root of the jacobian, for we need merely find points z_1, z_2, z_3 in the regions C_1, C_2, C_3 such that

$$(z_1, z_2, z_3, z_4) = \frac{p_1}{m}$$

and allow all the roots of the ground forms in each of those regions to coincide at those points. Any point of a region C_1, C_2, C_3 which is the envelope of more than one root of a ground form can be a position of pseudo-equilibrium and hence a root of the jacobian. If any of the regions C_1, C_2, C_3 is the envelope of merely one root of a ground form, then no point in that region but not in any other of the regions C_1, C_2, C_3, C_4 can be a position of equilibrium or of pseudo-equilibrium and hence no such point can be a root of the jacobian. If a point is common to two of the regions C_1, C_2, C_3, C_4 it is a point of C_4 and hence is a point of the envelope of the roots of the jacobian.

We have now proved the theorem except for its last sentence, to the demonstration of which we now proceed. When the roots of the ground forms in the regions C_1, C_2, C_3 coincide, the regions C_1, C_2, C_3, C_4 contain respectively the following numbers of roots of the jacobian: $m - 1, p_1 - m - 1, p_2 - 1, 1$. The roots of the jacobian vary continuously when the roots of the ground forms vary continuously; no root of the jacobian can enter or leave any of the regions C_1, C_2, C_3, C_4 which has no point in common with any other of those regions which is a part of the envelope of the roots of the jacobian.

The proof of Theorem III is now complete.* It applies to the sphere as well as the plane, since everything essential in the theorem is invariant under stereographic projection.

Instead of considering primarily the jacobian of two binary forms as heretofore, we may consider a rational function $f(z)$, introduce homogeneous coördinates, and compute the value of the derivative $f'(z)$ in terms of J, the jacobian of the binary forms which are the numerator and denominator of $f(z)$. We find that *the roots of $f'(z)$ are the roots of J and a double root at infinity, except that when one of these points is also a pole of $f(z)$ it cannot be a root of $f'(z)$*.† Application of Theorem III gives a theorem analogous to Theorem III, but which we state in a form slightly different from the statement of that theorem.

THEOREM. *If the circular regions C_1, C_2, C_3 contain respectively m roots (or poles) of a rational function $f(z)$ of degree p, all the remaining roots (or poles) of $f(z)$, and all the poles (or roots) of $f(z)$, then all the roots of $f'(z)$ lie in the regions C_1, C_2, C_3, and a fourth circular region C_4 determined as the envelope of points z_4 such that*

$$(z_1, z_2, z_3, z_4) = \frac{p}{m},$$

while the envelopes of z_1, z_2, z_3 are respectively C_1, C_2, C_3,—except that there are two roots at infinity if $f(z)$ has no pole there. Except for these two additional roots, if any of the regions C_i $(i = 1, 2, 3, 4)$ has no point in common with any other of those regions which contains a root of $f'(z)$, then that region contains the following number of roots of $f'(z)$ for $i = 1, 2, 3, 4$ respectively:

$$m - 1, \quad p - m - 1, \quad q_3 - 1, \quad 1;$$

or

$$q_1 - 1, \quad q_2 - 1, \quad p - 1, \quad 1,$$

according as C_1 contains m roots or m poles of $f(z)$; here q_i indicates the number of distinct poles of $f(z)$ in C_i.

Perhaps the following special cases of this theorem are worth stating explicitly.

If $f(z)$ is a rational function whose m_1 finite roots (or poles) lie on or within a circle C_1 with center α_1 and radius r_1 and whose m_2 finite poles (or roots) lie on or within a circle C_2 with center α_2 and radius r_2, and if $m_1 > m_2 > 0$, then

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* It may be noticed that this proof does not explicitly use the fact that C_4 is a circular region.

If C_1, C_2, C_3 are coaxial circles with no point in common, Theorem III reduces essentially to Theorem II (I, p. 294). If $m = 0$ or $p_1 - m = 0$, the regions C_1, C_2, and C_4 can be considered to coincide; this gives Theorem III (I, p. 296), which is due to Bôcher.

† See I, p. 297.

all the finite roots of $f'(z)$ lie in C_1, C_2, and a third circle C_3 whose center is

$$\frac{m_1 \alpha_2 - m_2 \alpha_1}{m_1 - m_2}$$

and radius

$$\frac{m_1 r_2 + m_2 r_1}{m_1 - m_2}.$$

*If $f(z)$ has no finite multiple poles, and if C_1, C_2, C_3 are mutually external, they contain respectively the following numbers of roots of $f'(z)$: $m_1 - 1$, $m_2 - 1$, 1. Under the given hypothesis, if $m_1 = m_2$ and if C_1 and C_2 are mutually external, these circles contain all the finite roots of $f'(z)$.**

If $f(z)$ is a polynomial m_1 of whose roots lie on or within a circle C_1 whose center is α_1 and radius r_1, and if the remaining m_2 roots lie on or within a circle C_2 whose center is α_2 and radius r_2, then all the roots of $f'(z)$ lie on or within C_1, C_2, and a third circle C_3 whose center is

$$\frac{m_1 \alpha_2 + m_2 \alpha_1}{m_1 + m_2}$$

and radius

$$\frac{m_1 r_2 + m_2 r_1}{m_1 + m_2}.$$

If these circles are mutually external, they contain respectively the following number of roots of $f'(z)$: $m_1 - 1$, $m_2 - 1$, 1.

If $f(z)$ is a polynomial of degree n with a k-fold root at P, and with the remaining $n - k$ roots in a circular region C, then all the roots of $f'(z)$ lie at P, in C, and in a circular region C' obtained by shrinking C toward P as center of similitude in the ratio $1 : k/n$. If C and C' have no point in common they contain respectively $n - k - 1$ roots and 1 root of $f'(z)$.†

A special case of this last theorem is the following

THEOREM. *If a circle includes all the roots of a polynomial $f(z)$, it also includes all the roots of $f'(z)$.*

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* A more restricted theorem than this has been proved not merely for rational functions but also for the quotient of two entire functions. See M. B. Porter, Proceedings of the National Academy of Sciences, vol. 2 (1916), pp. 247, 335.

There is no theorem analogous to the theorem of the present paper if $m_1 = m_2$ and if C_1 and C_2 are not mutually external. For we may consider all the roots and all the poles of $f(z)$ to coincide, so that $f(z)$ reduces to a constant and every point of the plane is a root of $f'(z)$.

† This theorem is true whether the circle C surrounds, passes through, or does not surround P, and whether the region C is interior or exterior to the circle C. The special case where P is the center of the circle C and the region C is external to that circle was pointed out in a footnote, I, p. 298. The special case where C does not surround P and the region C is interior to the circle C was pointed out to me by Professor D. R. Curtiss.

The latter theorem is equivalent to the well-known theorem of Lucas: *If all the roots of a polynomial $f(z)$ lie on or within any convex polygon, then all the roots of $f'(z)$ lie on or within that polygon.*

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